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A Remark on Generalized Covering Groups

Behrooz Mashayekhy

Department of Mathematics

Ferdowsi University of Mashhad

P.O.Box 1159-91775

Mashhad, Iran

E-mail: mashaf@science2.um.ac.ir

Abstract

Let $\mathcal{N}_c$ be the variety of nilpotent groups of class at most c $(c \geq 2)$ and $G = Z_r \oplus Z_s$ be the direct sum of two finite cyclic groups. It is shown that if the greatest common divisor of r and s is not one, then G does not have any $\mathcal{N}_c$-covering group for every $c \geq 2$. This result gives an idea that Lemma 2 of J.Wiegold [6] and Haebich's Theorem [1], a vast generalization of the Wiegold's Theorem, can *not* be generalized to the variety of nilpotent groups of class at most $c \geq 2$.

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1. Notation and Preliminaries

We assume that the reader is familiar with the notiona of the verbal subgroup, $V(G)$, and the marginal subgroup, $V^*(G)$, associated with a variety of groups, $\mathcal{V}$, and a group G , and the basic commutators (see [2]).

Let $\mathcal{V}$ be a variety of groups, and G be a group with the following free presentation , $1 \to R \to F \to G \to 1$. Then the *Baer-invariant* of G , with respect to the variety $\mathcal{V}$, denoted by $\mathcal{V}M(G)$, is defined to be $R \cap V(F)/[RV^*F]$ where $V(F)$ is the verbal subgroup of F and

$$[RV^*F] =< v(f_1, \dots, f_{i-1}, f_i r, f_{i+1}, \dots, f_n) v(f_1, \dots, f_i, \dots, f_n)^{-1} \mid$$

$$r \in R \ , \ f_i \in F, \ v \in V \ , \ 1 \le i \le n \ , \ n \in \mathbf{N} > \ .$$

In particular, if $\mathcal{V}$ is the variety of nilpotent groups of class at most c $(c \ge 1)$, then

$$\mathcal{V}M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, \ _cF]} \ .$$

A $\mathcal{V}$_stem cover of G is an exact sequence $1 \to A \to G^* \to G \to 1$ such that $A \subseteq V(G^*) \cap V^*(G^*)$,where G^* is said to be a $\mathcal{V}$-covering group of G . It is of interest to know which class of groups does *not* have a $\mathcal{V}$-covering group. For further details see C.R.Leedham-Green and S.McKay [4].

2. The Main Result

The following theorem is important in our study.

Theorem 1

Let $\mathcal{N}_c$ be the variety of nilpotent groups of class at most c $(c \geq 1)$ and r, s be two positive integers with $(r, s) = d$ their greatest common divisor. If $G = Z_r \oplus Z_s$, then $\mathcal{N}_c M(G) \cong Z_d \oplus Z_d \oplus \ldots \oplus Z_d$ (n-copies) , where n is the number of basic commutators of weight $c+1$ on two letters.

Proof.

Take the following free presentation for G :

$$1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1 \ ,$$

where F is the free group on $\{x_1, x_2\}$ and $R =< {x_1}^r, {x_2}^s, \gamma_2(F) >$. Clearly $R = S\gamma_2(F)$, where $S =< {x_1}^r, {x_2}^s >^F$. So the Baer-invariant of G with respect to the variety $\mathcal{N}_c$ is

$$\mathcal{N}_c M(G) = \frac{R \cap \gamma_{c+1}(F)}{[R, \ _cF]} = \frac{S\gamma_2(F) \cap \gamma_{c+1}(F)}{[s\gamma_2(F), \ _cF]}$$

$$= \frac{\gamma_{c+1}(F)}{[S, \ _cF]\gamma_{c+2}(F)} \cong \frac{\gamma_{c+1}(F)/\gamma_{c+2}(F)}{[S, \ _cF]\gamma_{c+2}(F)/\gamma_{c+2}(F)}$$

By P.Hall's Theorem (see[2]) $\gamma_{c+1}(F)/\gamma_{c+2}(F)$ is a free abelian group freely generated by all the basic commutators of weight $c+1$ on two letters. For all $a_i \in F$ and any $k \in \mathbf{Z}$ we have

$$[a_1, \ldots, {a_i}^k, \ldots, a_{c+1}] = [a_1, \ldots, a_i, \ldots, a_{c+1}]^k \qquad (mod \ \ \gamma_{c+2}(F)) \ .$$

Hence $[S,_c F]\gamma_{c+2}(F)/\gamma_{c+2}(F)$ is a free abelian group freely generated by the set of all d-th powers of all basic commutators of weight $c+1$ on two letters. Hence $\mathcal{N}_c M(G) \cong Z_d \oplus \ldots \oplus Z_d$ (n-copies) where n is the number of basic commutators of weight $c+1$ on two letters. $\Box$

The above theorem is somehow a generalization of M.R.R.Moghaddam's Theorem (see [5]).

Now we are in a position to state the main theorem.

Theorem 2

Let $\mathcal{N}_c$ be the variety of nilpotent groups of class at most c $(c \geq 2)$, and let $(r,s) = d \neq 1$ be the greatest common divisor of the positive integers r, s. If $G = Z_r \oplus Z_s$, then G does *not* admit any $\mathcal{N}_c$-stem cover for every $c \geq 2$ and so G has no $\mathcal{N}_c$-covering group.

Proof.

Assume by way of cotradiction that $1 \to A \to G^* \to G \to 1$ is an $\mathcal{N}_c$-stem cover for G . Thus $A \subseteq \gamma_{c+1}(G^*) \cap Z_c(G^*)$ (where $Z_c(G^*)$ is the c-th center of G^*) , $G^*/A \cong G$ and $A \cong \mathcal{N}_c M(G)$. Since G is abelian, $\gamma_2(G^*) \subseteq A$. Therefore $\gamma_{c+2}(G^*) = 1$, using the fact $\gamma_2(G^*) \subseteq Z_c(G^*)$. It implies that

$$1 = \gamma_{c+2}(G^*) = [\gamma_{c+1}(G^*), G^*] \supseteq [\gamma_2(G^*), G^*] = \gamma_3(G^*) \ .$$

As $c \geq 2$, we obtain $\gamma_{c+1}(G^*) = 1$ and hence $\mathcal{N}_c M(G) \cong A = 1$ which is a contradiction by Theorem 1 . $\Box$

Remark

(i) If $c=1$ then one may construct a covering group for G . (See G.Karpilovsky [3])

(ii) J.Wiegold in [6] showed that if G_1, G_2 are two finite groups and G_1^*, G_2^* any covering groups of G_1, G_2 , respectively, then the second nilpotent product $G_1^* \overset{2}{*} G_2^*$ is a covering group of $G_1 \times G_2$. Also W.Haebich constructed a covering group for a regular product of a class of groups (see [1]). Now our Theorem 2 shows that these notions can not be generalized to the variety of nilpotent groups of class $c \geq 2$. Because, in general, the direct product $G_1 \times G_2$ may not have an $\mathcal{N}_c$-covering group ($c \geq 2$).

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